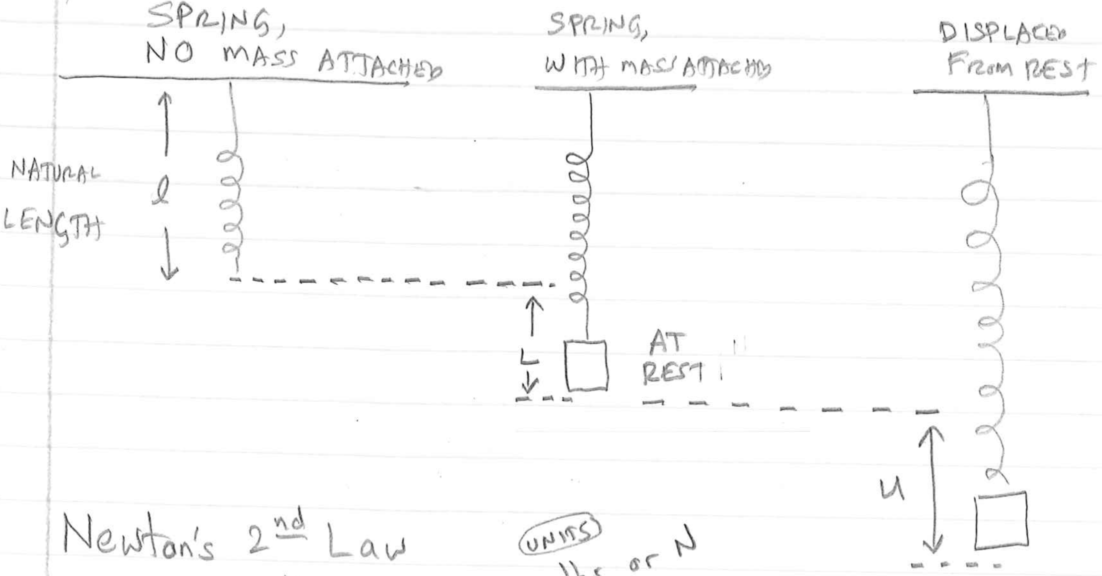
3.7: FREE VIBRATIONS (Simple Harmonic Motion)

ON SIDE BOARD!

OPEN ANIMATION



Newton's 2^{nd} Law

$$ma = F$$

(units) lbs or N → weight

Forces

① GRAVITY: $F_g = w = mg$ ← WILL HAVE POSITIVE = DOWN

mass (units) kg or $\frac{lbs}{(ft/sec^2)}$

	METRIC	IMPERIAL
MASS	kg	
g	$9.8\ m/s^2$	$32\ ft/s^2$
force	N	lbs
	NOTE: $N = 9.8 \cdot kg$	$MASS = \frac{lbs}{32}$

② HOOKE'S LAW: OPPOSITE DIRECTION PULLED

$$F_s = -k\underbrace{(L+u)}_{\text{DISTANCE FROM NATURAL LENGTH}} = -kL - ku$$

SPRING CONSTANT (FORCE TO HOLD 1 UNIT AWAY)

(units) $\frac{kg\,m/sec^2}{m} \to \frac{N}{m}$ or $\frac{lbs}{ft}$

③ DAMPING (FRICTION):

$$F_d = -\gamma u'$$

PROPORTIONAL TO VELOCITY (OPPOSITE DIRECTION)

(units) $\frac{N}{(m/s)}$ or $\frac{lbs}{(ft/s)}$

BIG NOTE

AT REST MEANS

$u=0$ AND $F_g + F_s = 0$

$\Rightarrow mg - kL = 0$

$\Rightarrow mg = kL$

$\Rightarrow \boxed{k = \frac{mg}{L}}$

THEREFORE

POSITIVE MASS, DAMPING, POSITIVE CONSTANT, SPRING CONSTANT

$$mu'' = \overbrace{mg - k(L}^{\to mg-kL=0}+u) - \gamma u' \Rightarrow \boxed{mu'' + \gamma u' + ku = 0}$$

POSITIVE

$\boxed{\text{UNDAMPED}}$ $\gamma = 0 \Rightarrow m u'' + ku = 0$

CHARACTERISTIC EQUATION : $e^{rt}(mr^2 + k) \overset{?}{=} 0$

$$\Rightarrow r^2 = -\frac{k}{m} \Rightarrow r_{1,2} = \pm\sqrt{\frac{k}{m}}\, i$$

COMPLEX ROOTS

$\lambda = 0, \quad \omega = \sqrt{\frac{k}{m}}$

SOL'N

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

ASIDE: STANDARD WAVE FORM

IDENTITY : $\underbrace{R\cos(\delta)}_{c_1}\cos(\omega t) + \underbrace{R\sin(\delta)}_{c_2}\sin(\omega t) = R\cos(\omega t - \overset{\phi}{\delta})$

$$c_1^2 + c_2^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta) = R^2(\cos^2(\delta) + \sin^2(\delta))$$
$$= R^2$$

$\Rightarrow R = \sqrt{c_1^2 + c_2^2}$ & $\left.\begin{aligned} \cos(\delta) &= \frac{c_1}{R} \\ \sin(\delta) &= \frac{c_2}{R} \end{aligned}\right\}$ δ = sol'n to this

$R = \sqrt{c_1^2 + c_2^2}$ = AMPLITUDE

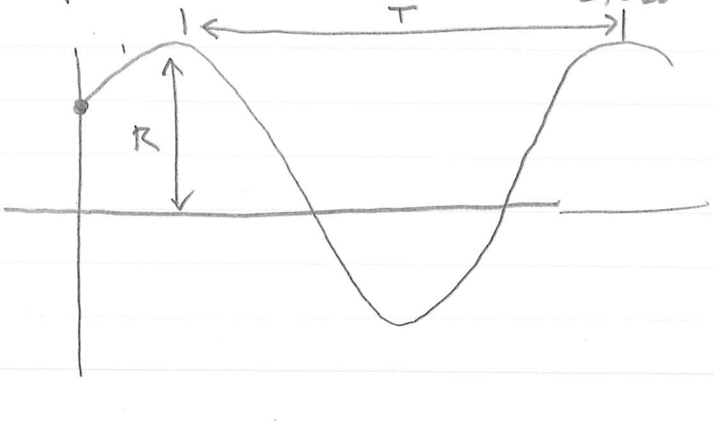
ω_0 = natural frequency $= \sqrt{\frac{k}{m}}$ $\quad \frac{\text{RAD}}{\text{SEC}}$

$f = \frac{\omega_0}{2\pi}$ = frequency in $\frac{\text{CYCLES}}{\text{SEC}}$

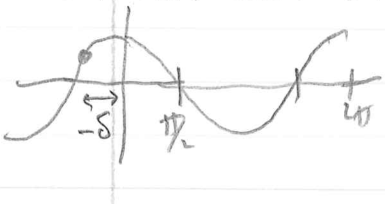
PERIOD

$T = \frac{1}{f} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$ = WAVELENGTH $\quad \frac{\text{SEC}}{\text{CYCLE}}$

δ = phase angle



STANDARD COSINE WAVE



m L

Ex) A $\frac{1}{2}$ kg object is placed on a spring and stretches it 2.45 meters beyond natural length (when at rest). Assume no damping. The mass is pulled downward 0.2 meters and given an initial downward velocity of 1 m/s. FIND AND DESCRIBE THE EQUATION FOR MOTION.

$m = \frac{1}{2}$ kg, $L = 2.45$ m

$N = kg \cdot m/s^2$

$$mg - kL = 0 \Rightarrow k = \frac{mg}{L} = \frac{\frac{1}{2} \cdot 9.8}{2.45} = 2 \ \frac{N}{m} \ \left(\frac{kg}{s^2}\right)$$

$$mu'' + \gamma u' + ku = 0$$

$$\frac{1}{2}u'' + 0u' + 2u = 0 \qquad u(0) = 0.2, \ u'(0) = 1$$

SOLVING: $\frac{1}{2}r^2 + 2 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$

$\lambda = 0, \ \omega = 2$

$$u = c_1\cos(2t) + c_2\sin(2t)$$

INITIAL CONDITIONS: $u(0) = 0.2 \Rightarrow c_1 = 0.2$

$u' = -2c_1\sin(2t) + 2c_2\cos(2t)$

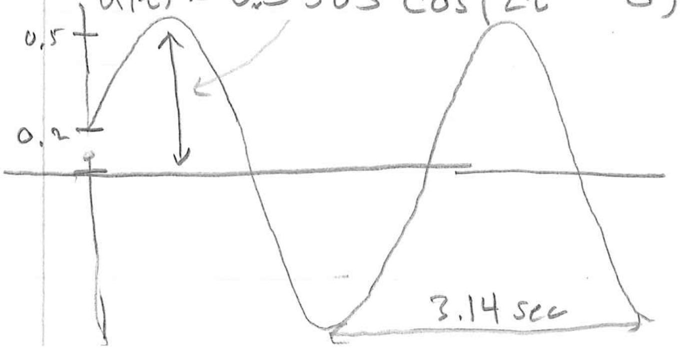
$u'(0) = 1 \Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} = 0.5$

$$\boxed{u(t) = 0.2\cos(2t) + 0.5\sin(2t)}$$

AMPLITUDE $= R = \sqrt{0.2^2 + 0.5^2} = \sqrt{0.29} \approx 0.5385$ m

$u(t) \approx 0.5385\cos(2t - \delta)$

$$\begin{cases} \cos(\delta) = \frac{0.2}{0.5385} \\ \sin(\delta) = \frac{0.2}{0.5385} \end{cases}$$



$\delta = \cos^{-1}\left(\frac{0.2}{0.5385}\right) \approx 1.19$ RAD

$T = \frac{2\pi}{2} = \pi$ SECONDS

DAMPED OSCILLATORS

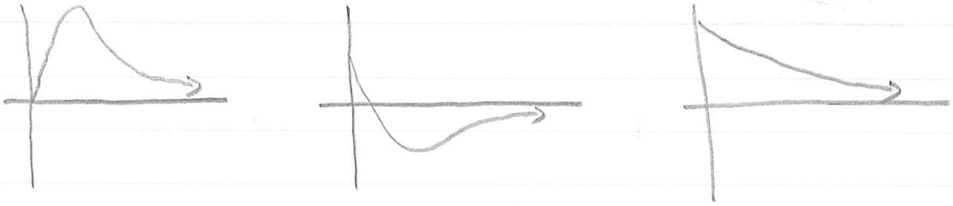
$$\overbrace{m u'' + \gamma u' + k u}^{\text{ALL POSITIVE NUMBERS}} = 0$$

$$\Rightarrow \quad r_{1,2} = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$$

$\boxed{1}$ If $\gamma^2 - 4mk > 0$ $\left(\gamma^2 > 4mk \, , \; \gamma > 2\sqrt{mk} \right)$ then we say the system is $\boxed{\text{OVERDAMPED}}$.

$\Rightarrow$ TWO NEGATIVE REAL ROOTS

$$\Rightarrow \quad u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$



DEPENDING ON INITIAL CONDITIONS WILL LOOK SORT OF LIKE ONE OF THESE

$\boxed{2}$ If $\gamma^2 - 4mk = 0 \Rightarrow \boxed{\text{CRITICALLY DAMPED}}$

$\Rightarrow$ ONE NEGATIVE ROOT $\quad r_{1,2} = -\frac{\gamma}{2m}$

$$\Rightarrow \quad u = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

WILL ALSO LOOK LIKE

$\boxed{3}$ IF $\gamma^2 - 4mk < 0$, $\boxed{\text{DAMPED OSCILLATIONS}}$

$$\lambda = -\frac{\gamma}{2m} \leftarrow \text{NEGATIVE}$$

$$\mu = \frac{1}{2m}\sqrt{4mk - \gamma^2} \leftarrow \text{Flipped to make positive}$$

$$= \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$$

OTHER WAYS TO WRITE THIS

$$\sqrt{\frac{\cancel{4}mk}{\cancel{4}m^2} - \frac{\gamma^2}{4m^2}}$$

$$= \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$$

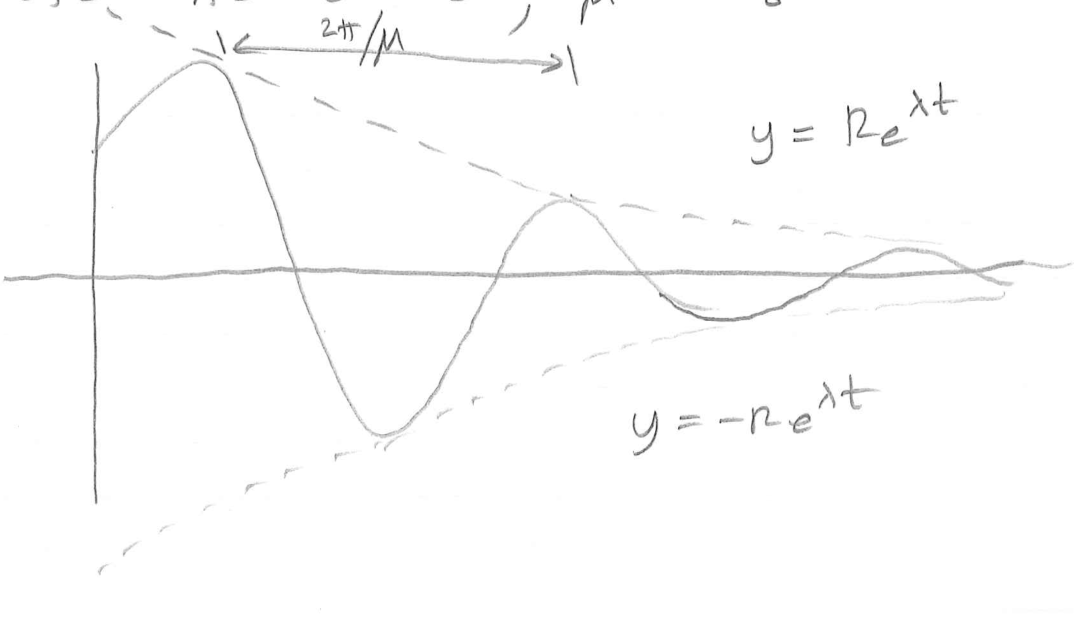
$$u = e^{\lambda t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right)$$

$$= \underbrace{Re^{\lambda t}}_{\text{Decreasing AMPLITUDE}} \cos(\mu t - \delta) \qquad \sqrt{c_1^2 + c_2^2} = R$$

μ = quasi-frequency $= \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$ ← NATURAL FREQUENCY WITH NO DAMPING

$T = \frac{2\pi}{\mu}$ = quasi-period

NOTE: AS $\gamma \to 0$, $\mu \to \omega_0$



Ex) A 4 lbs object stretches a spring 6 inches beyond natural length (at rest). A damping force of 2 lbs applies when velocity is 4 ft/sec. Assume initial displacement is 3 inches and it is released from rest (no initial velocity).

$\leftarrow \frac{3}{12} = \frac{1}{4}$ ft

FORCE!

$$mg = 4 \text{ lbs} \Rightarrow m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8} \quad \frac{\text{lbs}}{\text{ft}/s^2}$$

$$L = \frac{6}{12} = \frac{1}{2} \text{ ft} \quad \& \quad mg - kL = 0 \Rightarrow k = \frac{mg}{L} = \frac{4}{1/2} = 8 \quad \frac{\text{lbs}}{\text{ft}}$$

$$\underbrace{F_d}_{-2} = -\gamma \underset{4}{u'} \Rightarrow -2 = -4\gamma \Rightarrow \gamma = 1/2 \quad \frac{\text{lbs}}{\text{ft/sec}}$$

$$\boxed{\frac{1}{8} u'' + \frac{1}{2} u' + 8u = 0 \quad , u(0) = 1/4, \ u'(0) = 0}$$

$$\Rightarrow u'' + 4u' + 64u = 0 \qquad r_{1,2} = \frac{-4}{2} \pm \frac{1}{2}\sqrt{16 - 4(64)}$$

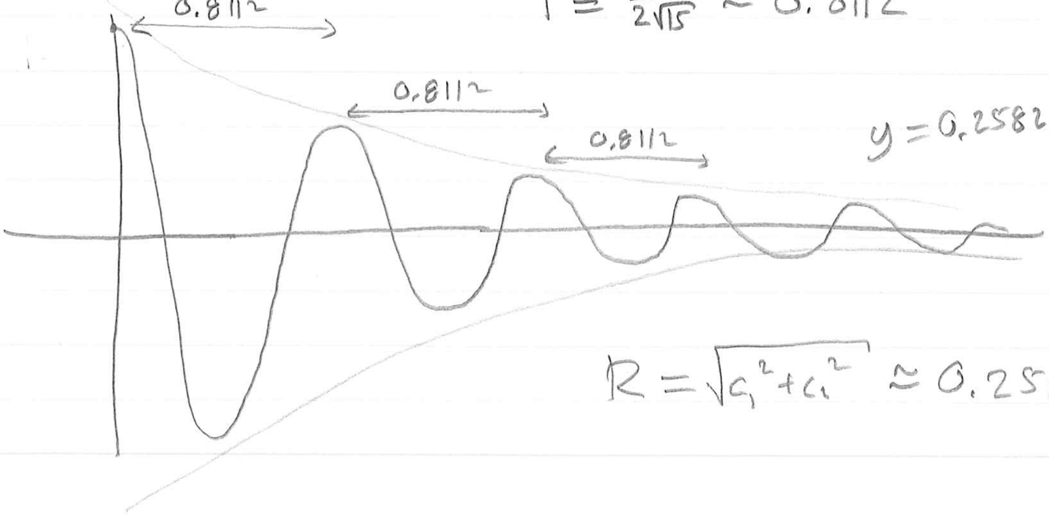
$\leftarrow 4\sqrt{15}\, i$

$$r_{1,2} = -2 \pm \frac{1}{2}\sqrt{-240}$$

$$= \underset{\lambda}{-2} \pm \underset{\mu}{2\sqrt{15}}\, i$$

$$u(t) = e^{-2t}\left(\overset{\frac{\sqrt{15}}{60}}{c_1} \cos\left(2\sqrt{15}\, t\right) + \overset{1/4}{C_2} \sin\left(2\sqrt{15}\, t\right)\right)$$

$$T = \frac{2\pi}{2\sqrt{15}} \approx 0.8112$$



$\delta = 0$

$$y = 0.2582 e^{-2t} \cos(\mu t - \overset{\downarrow}{\delta})$$

$$R = \sqrt{c_1^2 + c_1^2} \approx 0.25819$$